Arabic mathematics : forgotten brilliance?

Recent research paints a new picture of the debt that we owe to Arabic/Islamic mathematics. Certainly many of the ideas which were previously thought to have been brilliant new conceptions due to European mathematicians of the sixteenth, seventeenth and eighteenth centuries are now known to have been developed by Arabic/Islamic mathematicians around four centuries earlier. In many respects the mathematics studied today is far closer in style to that of the Arabic/Islamic contribution than to that of the Greeks.

There is a widely held view that, after a brilliant period for mathematics when the Greeks laid the foundations for modern mathematics, there was a period of stagnation before the Europeans took over where the Greeks left off at the beginning of the sixteenth century. The common perception of the period of 1000 years or so between the ancient Greeks and the European Renaissance is that little happened in the world of mathematics except that some Arabic translations of Greek texts were made which preserved the Greek learning so that it was available to the Europeans at the beginning of the sixteenth century.

That such views should be generally held is of no surprise. Many leading historians of mathematics have contributed to the perception by either omitting any mention of Arabic/Islamic mathematics in the historical development of the subject or with statements such as that made by Duhem in [3]:-

... *Arabic science only reproduced the teachings received from Greek science.*

Before we proceed it is worth trying to define the period that this article covers and give an overall description to cover the mathematicians who contributed. The period we cover is easy to describe: it stretches from the end of the eighth century to about the middle of the fifteenth century. Giving a description to cover the mathematicians who contributed, however, is much harder. The works [6] and [17] are on "Islamic mathematics", similar to [1] which uses the title the "Muslim contribution to mathematics". Other authors try the description "Arabic mathematics", see for example [10] and [11]. However, certainly not all the mathematicians we wish to include were Muslims; some were Jews, some Christians, some of other faiths. Nor were all these mathematicians Arabs, but for convenience we will call our topic "Arab mathematics".

The regions from which the "Arab mathematicians" came was centred on Iran/Iraq but varied with military conquest during the period. At its greatest extent it stretched to the west through Turkey and North Africa to include most of Spain, and to the east as far as the borders of China.

The background to the mathematical developments which began in Baghdad around 800 is not well understood. Certainly there was an important influence which came from the Hindu mathematicians whose earlier development of the decimal system and numerals was important. There began a remarkable period of mathematical progress with al-Khwarizmi's work and the translations of Greek texts.

This period begins under the Caliph Harun al-Rashid, the fifth Caliph of the Abbasid dynasty, whose reign began in 786. He encouraged scholarship and the first translations of Greek texts into Arabic, such as Euclid's *Elements* by al-Hajjaj, were made during al-Rashid's reign. The next Caliph, al-Ma'mun, encouraged learning even more strongly than his father al-Rashid, and he set up the House of Wisdom in Baghdad which became the centre for both the work of translating and of of research. Al-Kindi (born 801) and the three Banu Musa brothers worked there, as did the famous translator Hunayn ibn Ishaq.

We should emphasise that the translations into Arabic at this time were made by scientists and mathematicians such as those named above, not by language experts ignorant of mathematics, and the need for the translations was stimulated by the most advanced research of the time. It is important to realise that the translating was not done for its own sake, but was done as part of the current research effort. The most important Greek mathematical texts which were translated are listed in [17]:-

Of Euclid's works, the Elements, the Data, the Optics, the Phaenomena, and On Divisions were translated. Of Archimedes' works only two - Sphere and Cylinder and Measurement of the Circle - are known to have been translated, but these were sufficient to stimulate independent researches from the 9th to the 15th century. On the other hand, virtually all of Apollonius's works were translated, and of Diophantus and Menelaus one book each, the Arithmetica and the Sphaerica, respectively, were translated into Arabic. Finally, the translation of Ptolemy's Almagest furnished important astronomical material.

The more minor Greek mathematical texts which were translated are also given in [17]:-

... Diocles' treatise on mirrors, Theodosius's Spherics, Pappus's work on mechanics, Ptolemy's Planisphaerium, and Hypsicles' treatises on regular polyhedra (the so-called Books XIV and XV of Euclid's Elements) ...

Perhaps one of the most significant advances made by Arabic mathematics began at this time with the work of al-Khwarizmi, namely the beginnings of algebra. It is important to understand just how significant this new idea was. It was a revolutionary move away from the Greek concept of mathematics which was essentially geometry.

Algebra was a unifying theory which allowed rational numbers, irrational numbers, geometrical magnitudes, etc., to all be treated as "algebraic objects". It gave mathematics a whole new development path so much broader in concept to that which had existed before, and provided a vehicle for future development of the subject. Another important aspect of the introduction of algebraic ideas was that it allowed mathematics to be applied to itself in a way which had not happened before. As Rashed writes in [11] (see also [10]):-

Al-Khwarizmi's successors undertook a systematic application of arithmetic to algebra, algebra to arithmetic, both to trigonometry, algebra to the Euclidean theory of numbers, algebra to geometry, and geometry to algebra. This was how the creation of polynomial algebra, combinatorial analysis, numerical analysis, the numerical solution of equations, the new elementary theory of numbers, and the geometric construction of equations arose.

Let us follow the development of algebra for a moment and look at al-Khwarizmi's successors. About forty years after al-Khwarizmi is the work of al-Mahani (born 820), who conceived the idea of reducing geometrical problems such as duplicating the cube to problems in algebra. Abu Kamil (born 850) forms an important link in the development of algebra between al-Khwarizmi and al-Karaji. Despite not using symbols, but writing powers of x in words, he had begun to understand what we would write in symbols as $x^n.x^m = x^{m+n}$. Let us remark that symbols did not appear in Arabic mathematics until much later. Ibn al-Banna and al-Qalasadi used symbols in the 15th century and, although we do not know exactly when their use began, we know that symbols were used at least a century before this.

Al-Karaji (born 953) is seen by many as the first person to completely free algebra from geometrical operations and to replace them with the arithmetical type of operations which are at the core of algebra today. He was first to define the monomials x, x^2, x^3, ... and $1/x$, $1/x^2$, $1/x^3$, ... and to give rules for products of any two of these. He started a school of algebra which flourished for several hundreds of years. Al-Samawal, nearly 200 years later, was an important member of al-Karaji's school. Al-Samawal (born 1130) was the first to give the new topic of algebra a precise description when he wrote that it was concerned:-

... with operating on unknowns using all the arithmetical tools, in the same way as the arithmetician operates on the known.

Omar Khayyam (born 1048) gave a complete classification of cubic equations with geometric solutions found by means of intersecting conic sections. Khayyam also wrote that he hoped to give a full description of the algebraic solution of cubic equations in a later work [18]:-

If the opportunity arises and I can succeed, I shall give all these fourteen forms with all their branches and cases, and how to distinguish whatever is possible or impossible so that a paper, containing elements which are greatly useful in this art will be prepared.

Sharaf al-Din al-Tusi (born 1135), although almost exactly the same age as al-Samawal, does not follow the general development that came through al-Karaji's school of algebra but rather follows Khayyam's application of algebra to geometry. He wrote a treatise on cubic equations, which [11]:-

... represents an essential contribution to another algebra which aimed to study curves by means of equations, thus inaugurating the beginning of algebraic geometry.

Let us give other examples of the development of Arabic mathematics. Returning to the House of Wisdom in Baghdad in the 9th century, one mathematician who was educated there by the Banu Musa brothers was Thabit ibn Qurra (born 836). He made many contributions to mathematics, but let us consider for the moment consider his contributions to number theory. He discovered a beautiful theorem which allowed pairs of amicable numbers to be found, that is two numbers such that each is the sum of the proper divisors of the other. Al-Baghdadi (born 980) looked at a slight variant of Thabit ibn Qurra's theorem, while al-Haytham (born 965) seems to have been the first to attempt to classify all even perfect numbers (numbers equal to the sum of their proper divisors) as those of the form $2^{k-1}(2^k - 1)$ where $2^k - 1$ is prime.

Al-Haytham, is also the first person that we know to state Wilson's theorem, namely that if p is prime then $1+(p-1)!$ is divisible by p. It is unclear whether he knew how to prove this result. It is called *Wilson's theorem* because of a comment made by Waring in 1770 that John Wilson had noticed the result. There is no evidence that John Wilson knew how to prove it and most certainly Waring did not. Lagrange gave the first proof in 1771 and it should be noticed that it is more than 750 years after al-Haytham before number theory surpasses this achievement of Arabic mathematics.

Continuing the story of amicable numbers, from which we have taken a diversion, it is worth noting that they play a large role in Arabic mathematics. Al-Farisi (born 1260) gave a new proof of Thabit ibn Qurra's theorem, introducing important new ideas concerning factorisation and combinatorial methods. He also gave the pair of amicable numbers 17296, 18416 which have been attributed to Euler, but we know that these were known earlier than al-Farisi, perhaps even by Thabit ibn Qurra himself. Although outside our time range for Arabic mathematics in this article, it is worth noting that in the 17^{th} century the Arabic mathematician Mohammed Baqir Yazdi gave the pair of amicable number 9,363,584 and 9,437,056 still many years before Euler's contribution.

Let us turn to the different systems of counting which were in use around the 10^{th} century in Arabic countries. There were three different types of arithmetic used around this period and, by the end of the 10^{th} century, authors such as al-Baghdadi were writing texts comparing the three systems.

1. Finger-reckoning arithmetic.
This system derived from counting on the fingers with the numerals written entirely in words; this finger-reckoning arithmetic was the system used by the business community. Mathematicians such as Abu'l-Wafa (born 940) wrote several treatises using this system. Abu'l-Wafa himself was an expert in the use of Indian numerals but these:-

... did not find application in business circles and among the population of the Eastern Caliphate for a long time.

Hence he wrote his text using finger-reckoning arithmetic since this was the system used by the business community.

2. Sexagesimal system.
The second of the three systems was the sexagesimal system, with numerals denoted by letters of the Arabic alphabet. It came originally from the Babylonians and was most frequently used by the Arabic mathematicians in astronomical work.

3. Indian numeral system.
The third system was the arithmetic of the Indian numerals and fractions with the decimal place-value system. The numerals used were taken over from India, but there was not a standard set of symbols. Different parts of the Arabic world used slightly different forms of the numerals. At first the Indian methods were used by the Arabs with a dust board. A dust board was needed because the methods required the moving of numbers around in the calculation and rubbing some out as the calculation proceeded. The dust board allowed this to be done in the same sort of way that one can use a blackboard, chalk and a blackboard eraser. However, al-Uqlidisi (born 920) showed how to modify the methods for pen and paper use. Al-Baghdadi also contributed to improvements in the decimal system.

It was this third system of calculating which allowed most of the advances in numerical methods by the Arabs. It allowed the extraction of roots by mathematicians such as Abu'l-Wafa and Omar Khayyam (born 1048). The discovery of the binomial theorem for integer exponents by al-Karaji (born 953) was a major factor in the development of numerical analysis based on the decimal system. Al-Kashi (born 1380) contributed to the development of decimal fractions not only for approximating algebraic numbers, but also for real numbers such as π. His contribution to decimal fractions is so major that for many years he was considered as their inventor. Although not the first to do so, al-Kashi gave an algorithm for calculating *n*th roots which is a special case of the methods given many centuries later by Ruffini and Horner.

Although the Arabic mathematicians are most famed for their work on algebra, number theory and number systems, they also made considerable contributions to geometry, trigonometry and mathematical astronomy. Ibrahim ibn Sinan (born 908), who introduced a method of integration more general than that of Archimedes, and al-Quhi (born 940) were leading figures in a revival and continuation of Greek higher geometry in the Islamic world. These mathematicians, and in particular al-Haytham, studied optics and investigated the optical properties of mirrors made from conic sections. Omar Khayyam combined the use of trigonometry and approximation theory to provide methods of solving algebraic equations by geometrical means.

Astronomy, time-keeping and geography provided other motivations for geometrical and trigonometrical research. For example Ibrahim ibn Sinan and his grandfather Thabit ibn Qurra both studied curves required in the construction of sundials. Abu'l-Wafa and Abu Nasr Mansur both applied spherical geometry to astronomy and also used formulas involving sin and tan. Al-Biruni (born 973) used the sin formula in both astronomy and in the calculation of longitudes and latitudes of many cities. Again both astronomy and geography motivated al-Biruni's extensive studies of projecting a hemisphere onto the plane.

Thabit ibn Qurra undertook both theoretical and observational work in astronomy. Al-Battani (born 850) made accurate observations which allowed him to improve on Ptolemy's data for the sun and the moon. Nasir al-Din al-Tusi (born 1201), like many other Arabic mathematicians, based his theoretical astronomy on Ptolemy's work but al-Tusi made the most significant development of Ptolemy's model of the planetary system up to the development of the heliocentric model in the time of Copernicus.

Many of the Arabic mathematicians produced tables of trigonometric functions as part of their studies of astronomy. These include Ulugh Beg (born 1393) and al-Kashi. The construction of astronomical instruments such as the astrolabe was also a speciality of the Arabs. Al-Mahani used an astrolabe while Ahmed (born 835), al-Khazin (born 900), Ibrahim ibn Sinan, al-Quhi, Abu Nasr Mansur (born 965), al-Biruni, and others, all wrote important treatises on the astrolabe. Sharaf al-Din al-Tusi (born 1201) invented the linear astrolabe. The importance of the Arabic mathematicians in the development of the astrolabe is described in [17]:-

The astrolabe, whose mathematical theory is based on the stereographic projection of the sphere, was invented in late antiquity, but its extensive development in Islam made it the pocket watch of the medievals. In its original form, it required a different plate of horizon coordinates for each latitude, but in the 11th *century the Spanish Muslim astronomer az-Zarqallu invented a single plate that worked for all latitudes. Slightly earlier, astronomers in the East had experimented with plane projections of the sphere, and al-Biruni invented such a projection that could be used to produce a map of a hemisphere. The culminating masterpiece was the astrolabe of the Syrian Ibn ash-Shatir* (1305-75), *a mathematical tool that could be used to solve all the standard problems of spherical astronomy in five different ways.*

Article by: *J J O'Connor* and *E F Robertson*

Bron: http://www-history.mcs.st-andrews.ac.uk/HistTopics/Arabic_mathematics.html